Exercise 64

To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 63 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

Solution

Start with the identity,

$$\sin a = \sin a.$$

Rewrite the left side.

$$(\sin a) \cdot 1 + (\cos a) \cdot 0 = \sin a$$

Use the formulas in (6) on page 119.

$$(\sin a) \cdot \lim_{h \to 0} \cos h + (\cos a) \cdot \lim_{h \to 0} \sin h = \sin a$$

$\sin a$ and $\cos a$ are constants and can be brought inside the respective limits.

$$\lim_{h \to 0} \sin a \cos h + \lim_{h \to 0} \cos a \sin h = \sin a$$

The limit of a sum is the sum of the limits.

$$\lim_{h \to 0} (\sin a \cos h + \cos a \sin h) = \sin a$$

Use the angle addition formula for sine.

$$\lim_{h \to 0} \sin(h + a) = \sin a$$

Therefore, sine is a continuous function.